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Optimality of Impulse Harvesting Policies

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Abstract We explore the link between cyclical and smooth resource exploitation. We define an impulse control framework which can generate both cyclical solutions and steady state solutions. For the cyclical solution, we establish a link with the discrete-time model by Dawid and Kopel (1997). For the steady state solution, we explore the relation to Clark's (1976) continuous control model. Our model can admit convex and concave profit functions and allows the integration of different stock dependent cost functions. We show that the strict concavity of the profit function is only a special case of a more general condition, related to submodularity, that ensures the existence of optimal cyclical policies.

1 Introduction

There exist two main types of harvesting policies for renewable resources such as animal or plant populations. The first type of policy is the smooth policy. In a continuous time model, at each point in time, an infinitely small part of

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the population is captured so that the size of the population never changes abruptly although the time derivative of the population size may be discontinuous. Numerous examples of such policies have been given in the pioneering work of Clark and Munro (1975) (see also Clark (1976)) for fisheries. The well-known harvesting policy of Faustmann (1849) (see also Johansson and Löfgren (1985)) for a balanced forest also belongs to this type: only the trees having reached the optimal felling age are cut. Although for each tree cohort the policy is an abrupt one, for the forest as a whole such a policy is a smooth one.

At the other extreme of the spectrum an impulse policy consists in harvesting some significant part of the population at some points in time while leaving the population to evolve in its natural environment between any two consecutive harvest dates. An example is again Faustmann's optimal cutting policy but now for single, even-aged, forest stands.

At an aggregate level, optimal impulse policies are quite rare for two main reasons. The first is that renewable resources are generally scattered all over the world with specific characteristics so that synchronized impulse harvesting of so many sources is unlikely. The second reason is that an aggregate impulse policy would induce hikes in the price path, thus opening the door for arbitrage opportunities when stockpiling costs are high. The arbitrage possibility stems from the very fact that stockpiling costs are nil for the resources left unexploited. As a result, the price hikes may be arbitraged by moderately changing the harvest date at a low opportunity cost. However at a micro level such impulse policies may be optimal, that is, profit maximizing strategies.

We propose in this paper a model of renewable resource management based on the impulse control framework (*cf.* Vind (1967), Léonard and Long (1998) or Seierstaed and Sydsaeter (1987)). This model generalizes previous discrete-time models and contains, as a limit, the classical continuous-time singular control model. We adopt very weak assumptions on the growth function and on the profit function which is allowed to depend on both the current stock and the size of the harvest. In particular, we do not impose any type of concavity. We characterize the solution to this problem by reducing it to two coupled optimization problems with two variable each. We are then able to discuss under which conditions the optimal trajectory exhibits *cycles* or not.

Cycles in deterministic models may occur for various reasons. The presence of state variables in addition to the state of the resource is a well-documented reason, both for discrete-time and continuous-time models: see for instance Benhabib and Nishimura (1985), Wirl (1995) and Feichtinger and Sorger (1996). The focus of the present paper is on one-dimensional models, where the existence of cycles results from other phenomena than "hidden" variables.

In discrete-time, one-dimensional models, cycles occur when optimal trajectories are not stationary. Benhabib and Nishimura (1985) have shown that such cycles occur under the assumption of concavity and submodularity of the profit function, plus additional technical assumptions. Olson and Roy (1996) show that concavity and supermodularity of that function implies the absence of cycles. On the other hand, Dawid and Kopel (1997; 1999) showed that a strictly convex gain function depending only on the capture may lead to optimal cyclical solutions. In the literature on one-dimensional continuous-time control models, cycles may also appear. Indeed, Lewis and Schmalensee (1977; 1979) found that cycles can be optimal in presence of increasing returns to scale, stock effects and modest re-entry costs. Liski et al. (2001) demonstrated the occurrence of cycles in a model with increasing returns to scale and modest adjustment costs, in the absence of stock effects.

Finally, note that in continuous-time models, the relevance of impulse control has been pointed out early in the literature, see Clark (1976, p. 58) where it is suggested that optimal policies may consist in one impulse followed by a continuous, smooth control. Early empirical evidence in the fisheries sector was provided by Hannesson (1975). On the other hand, the cutting policy of Faustmann's is based on an impulse control with cycles.

We show that the conditions for the existence of cyclical solutions involve a close combination of the growth function and the cost function, thereby emphasizing that the convexity of the cost function, or its dependence on the stock level, are not the only issues worth considering. We then discuss how a Clark-like steady-state solution emerges as a limit of small and frequent harvest operations in our model. We also show that we can reproduce and generalize Dawid and Kopel's results, although the latter were obtained with a discrete-time model and without stock effects.

The article is structured as follows. We present the impulse control problem in section 2, we characterize the type of solution in section 3 and the optimal cycle in section 4. We then establish the link to Clark's continuous control solution and to Dawid and Kopel's discrete control model in section 5. The last section is devoted to the conclusion.

2 The impulse control model

2.1 The Model

The resource dynamics

We consider a renewable resource, for which dynamics, in the absence of any harvest, is given by:

$$\dot{x}(t) \;=\; F(x(t)) \;, \quad t \geq 0, \tag{1}$$

where $x(t)$ is the size of the population at time t and F, stationary through time, is the growth rate function. The function F is assumed to satisfy the following conditions.

Assumption 1 There exist numbers x_{sup} and x_s, $0 < x_s < x_{sup} \leq +\infty$, such that the function $F : (0, x_{sup}) \to \mathbb{R}$ is positive over the interval $(0, x_s)$ and negative over the interval (x_s, x_{sup}), with $F(0) = F(x_s) = 0$, where $\lim_{x\downarrow 0} F(x) = F(0)$, and $\lim_{x\uparrow x_{sup}} F(x) = -\infty$. The function F is measurable and bounded above. It is assumed that the differential equation (1) admits a unique solution for every initial stock $x_0 \in (0, x_{sup})$.

The population level x_s is the standard long-run carrying capacity of the environment to which, absent any catch, the population is converging for any x_0 such that $0 < x_0 < x_{sup}$. Note that the assumptions on F are very weak, specially the monotonicity assumptions. For instance, F is not necessarily concave, and may have several local maxima. Continuity over $(0, x_{sup})$ is not required either, as long as (1) admits a unique solution.

The harvesting process

We are interested in the optimal exploitation of this resource by a discrete harvest process, i.e. within the framework of impulse control models.[1]

Accordingly, we define an impulse exploitation policy IP $:= \{(t_i, I_i), i = 1, 2, \ldots\}$ as a sequence of harvesting dates t_i and instantaneous harvests I_i, one for each date. The sequence of dates may be empty, finite or infinite. It is such that $0 \leq t_1$, and $t_i \leq t_{i+1}$, $i = 1, 2, \ldots$ and $\lim_{i \to +\infty} t_i = +\infty$. By convention, we shall assume that if the sequence is finite with $n \geq 0$ values, then $t_i = +\infty$ for all $i > n$.

The sequence of harvests must satisfy:

$$I_i \geq 0 \quad \text{and} \quad x_i - I_i \geq 0 \ , \tag{2}$$

where

$$x_i \ = \ \lim_{t \uparrow t_i} x(t) \ , \quad \text{with } x_1 = x_0 \text{ given if } t_1 = 0, \tag{3}$$

and such that the following constraints hold:

$$\dot{x}(t) = F(x(t)) \text{ for } t_i < t < t_{i+1} \text{ with } x(t_i) = x_i - I_i, \quad i = 1, 2, \ldots \tag{4}$$

$$\dot{x}(t) = F(x(t)) \text{ for } 0 < t < t_1 \text{ with } x(0) = x_0 \quad \text{if } t_1 > 0. \tag{5}$$

In other words: x_i is the size of the population just before the harvesting date t_i, and $x_i - I_i$ its size just after that same date. If $t_1 = 0$, the population x_1 is supposed to be inherited from the past, and denoted by x_0. Harvests cannot be negative nor exceed the population size. The conditions (2)–(5) define the set of *feasible* IPs, denoted by $\mathcal{F}_{x_0}$.

[1] Impulse control policies in infinite horizon consist in an unbounded sequence of decisions. For the discussion of impulse control models, see for example Léonard and Long (1998), Seierstaed and Sydsaeter (1987).

The harvester's profits

Monetary profits generated by any harvest depend upon the size of the catch and the size of the population at the catching time. We assume that the profit function is stationary through time so that whatever t_i, I_i and x_i, the current profits at time t_i amount to $\pi(x_i, I_i)$.[2] The profit function is assumed to have the following standard properties.

Assumption 2 The function $\pi(x, I)$ is defined on the domain $\mathcal{D} := \{(x, I), x \in (0, x_{sup}), I \in [0, x]\}$. It is of class C^1, positive and bounded, and such that $\pi(x, 0) = 0$, $\forall x \in (0, x_{sup})$. The derivative $\pi_I(x, I) := (\partial\pi/\partial I)(x, I)$ admits a finite limit when $I \downarrow 0$ for all $x \in (0, x_{sup})$.

Profits are discounted using a constant instantaneous discount rate, denoted by r, $r > 0$.

The manager's problem is to choose some policy maximizing the sum of the discounted profits, that is to solve the problem (P):

$$\text{(P)} \qquad \sup_{\mathrm{IP} \in \mathcal{F}_{x_0}} \Pi(\mathrm{IP}) \ := \ \sum_{i=1}^{\infty} e^{-rt_i} \ \pi(x_i, I_i) \ .$$

The function Π is assumed to be well defined over the whole set $\mathcal{F}_{x_0}$.[3]

Approximation of a continuous control

The classical modeling of a controlled renewable resource involves the modified dynamics

$$\dot{x}(t) \ = \ F(x(t)) \ - \ h(t) \ ,$$

where $h(t)$ is the rate of harvest at time t. The harvester's profit is some instantaneous profit function $p(x, h)$ depending on the current stock and the rate of extraction. It is possible to approximate the trajectories of a continuously controlled system by an impulse-controlled one. For instance, by choosing the impulses so that the two trajectories are periodically synchronized, say, every δt units of time. When the period δt tends to 0, the distance between trajectories should go to 0. The gain of such a "micro-impulse" policy can be estimated as follows:[4] during the interval $[t_0, t_0 + \delta t]$, the resource under the dynamics (1) goes from x to $x + \delta t F(x) + o(\delta t)$. The controlled resource goes from x to $x + \delta t(F(x) - h(t_0)) + o(\delta t)$. The discrepancy is corrected with an impulse of $I = \delta t \times h(t_0)$. According to Assumption 2, we have $\pi(x, 0) = 0$ for all x,

[2] Thus we assume that the resource stock per se is not generating any surplus flow as in Hartman (1976), Smith (1977) and Berck (1981) to quote a few pioneering works along this way. This effect can be neglected for a wide spectrum of renewable resources. For example, most fisheries do not generate such surplus.

[3] Observe that we formulate our problem with a "sup" and not a "max" because we are interested in the possibility that the maximum is not reached inside the set $\mathcal{F}_{x_0}$.

[4] We do not pursue here the task of formally proving these claims, since this is not essential for the rest of the analysis.

which implies that $\pi_x(x,0) = 0$ also for all x. Therefore, the impulse generates a gain of:

$$\pi(x + \delta t \times F(x), \delta t \times h(t_0)) \;=\; (\delta t \times h(t_0))\, \pi_I(x,0) \;+\; o(\delta t)\;.$$

In the limit, the gain obtained by the series of impulses is the same as the continuously accumulated gain with profit function $p(x,h) = h\pi_I(x,0)$. This function is of the specific form used in the singular control model of Clark. We come back to this property in Section 5.1.

2.2 The Dynamic Programming Principle

We use the Dynamic Programming approach to solve the problem. The following theorem insures the existence of a unique value for the problem.

Theorem 1 *The value function*

$$v(x) \;=\; \sup_{IP \in \mathcal{F}_x} \Pi(IP) \tag{6}$$

is the unique solution of the following variational equation:

$$v(x) \;=\; \sup_{\substack{t\geq 0\\ 0\leq y\leq \phi(t,x)}} \; e^{-rt}\left[\pi(\phi(t,x), \phi(t,x) - y) + v(y)\right]\;, \tag{7}$$

where $\phi(t,x)$ is the trajectory of the system at time t, solution of the dynamics (1) *with $x(0) = x$.*

For a standard proof of this dynamic programming result, see (Davis, 1993, Theorem (54.19), page 236).

3 Reduction to Cyclical Policies

In this section we investigate the impulse control model and propose an approach for characterizing its solutions. Our approach is to determine the structure of solutions under the quite general assumptions of the previous section. The price to pay for this generality is that our results do not guarantee the uniqueness of solutions, which must be examined on a case-by-case basis.

Our line of argument will be the following. First of all, the Dynamic Programming principle implies that, under any optimal policy for Problem (P), if the stock reaches some level already attained in the past, the action chosen in the past (to harvest or not to harvest) should still be optimal. This mere fact combined with the positive growth of the stock's natural dynamics tends to select policies that are *cyclical* in the sense that they let the stock grow to some level, harvest it down so some other level, and repeat. However, it may still be that under the optimal policy, the stock never reaches twice the same level. We show that when the gain function has a certain *submodularity* property, such trajectories cannot be optimal. Optimal policies are therefore essentially cyclical. Moreover, joining the optimal cycle must be done with at most one harvest.

The optimization problem is then reduced to finding: a) what is the optimal cycle; b) what is the optimal way to reach the optimal cycle from a given initial stock. Finding the optimal cycle is a relatively simple optimization problem which we call the "Auxiliary Problem". But the solution to this problem may correspond to degenerate cycles, which we interpret as continuous harvesting policies *à la* Clark. We show in the next section that the submodularity assumption is again the key to determine whether the optimal cycle is a true cycle or a degenerate one.

We proceed now with the definitions and the precise statements of these principles.

3.1 Cyclical Policies and the Auxiliary Problem

Cyclical policies A cyclical policy has two components: a cycle which is characterized by two values $\underline{x}$ and $\bar{x}$ with $\underline{x} < \bar{x}$, or equivalently by an interval $[\underline{x}, \bar{x}]$; and a transitory part which describes how the trajectory evolves from the initial stock to the cycle. The transitory part consists in a finite (possibly empty) sequence of harvests, such that, after the last harvest, the remaining population is less than $\bar{x}$. We first concentrate on the cycle.

Hence, a cycle has two main parameters, which are such that $0 \leq \underline{x} < \bar{x} \leq x_s$.[5] When in its cyclical part, a policy acts as follows: a) let the population grow to $\bar{x}$; b) harvest $\bar{x} - \underline{x}$; and repeat. Such a policy applies only to initial populations $x_0 \leq \bar{x}$. In other words, the transitory part can be dispensed with only for such an initial population.

Gain under a cyclical policy We will denote by $G(\underline{x}, \bar{x}, x_0)$ the value of discounted profits in a policy without the transitory part, applied to an initial population of x_0. The complete definition of the function G involves several cases, corresponding to the limit cases for $\bar{x}$ and $\underline{x}$.

It is convenient to define the function $\tau(x, y)$ as the time necessary for the dynamics to go from value x to y, $x \leq y$. It turns out that for all $0 < x \leq y < x_s$:

$$\tau(x, y) \;=\; \int_x^y \frac{1}{F(u)}\, du. \tag{8}$$

Since, by Assumption 1, $F(x_s) = 0$, the integral defining $\tau(x, y)$ is singular when $y = x_s$. The limit when $y \to x_s$ may therefore be finite or infinite, depending on the function F. Another feature of Assumption 1 is that $F(0) = 0$. Consequently, if $x(0) = 0$, a solution to the dynamics (1) is $x(t) = 0$ for all $t \geq 0$. This implies the convention that $\tau(0, y) = +\infty$ if $y > 0$, and $\tau(0, 0) = 0$.[6]

[5] Since $\bar{x}$ represents the population level until which the resource grows before harvesting, there is no point in considering $\bar{x} > x_s$ since the population cannot grow to such a level.

[6] This convention does not mean that $\lim_{x \downarrow 0} \tau(x, y) = +\infty$ in every situation.

The value of the total profit function G can be expressed as:

i) If $0 \leq \underline{x} < \bar{x} \leq x_s$:

$$G(\underline{x}, \bar{x}, x_0) := \pi(\bar{x}, \bar{x} - \underline{x}) \frac{e^{-r\tau(x_0,\bar{x})}}{1 - e^{-r\tau(\underline{x},\bar{x})}} . \tag{9}$$

The convention is that: if $\underline{x} = 0$, the term $\exp(-r\tau(\underline{x}, \bar{x}))$ should be replaced by 0. Likewise, $\exp(-r\tau(\underline{x}, \bar{x}))$ and $\exp(-r\tau(x_0, \bar{x}))$ are 0 if $\bar{x} = x_s$ and $\lim_{y\to x_s} \tau(x, y) = +\infty$.

ii) For $\underline{x} = \bar{x}$, Assumption 2 allows to define G by continuity as:

$$G(x, x, x_0) = \pi_I(x, 0) \; \frac{F(x)}{r} \; e^{-r\tau(x_0,x)} . \tag{10}$$

For the cases $\underline{x} = \bar{x}$, the value G is not that of a well-defined impulse control policy. As we have seen in Section 2.1, this value is that of a *continuous* harvesting policy, which can be seen as a degenerate impulse policy. The harvest rate of this continuous policy is constant and equal to $F(x)$.

Finally, by using the fact that $\tau(x, y)$ defined in (8) is also defined for $y \leq x$, expressions (9) and (10) provide values for the function G when $x_0 > \bar{x}$ as well. Of course, these situations do not correspond to an implementable harvesting policy, and the function loses its economic meaning. In subsection 3.3 we will study the transitory part of a cyclical policy for which the case $x_0 > \bar{x}$ has an economic meaning.

The auxiliary problem

Having defined the function $G(\underline{x}, \bar{x}, x_0)$ for all $0 \leq \underline{x} \leq \bar{x} \leq x_s$ and all $0 \leq x_0 \leq x_s$, we now define the auxiliary problem (AP):

$$(\mathrm{AP}): \quad \max_{\underline{x},\, \bar{x};\; 0\leq\underline{x}\leq\bar{x}\leq x_s} G(\underline{x}, \bar{x}, x_0).$$

Under Assumption 2 it turns out that G is lower semi-continuous as a function of $(\underline{x}, \bar{x})$. The problem (AP) has therefore always a solution. For the purpose of the discussion to come, it is important to distinguish the case where the solution is such that $\underline{x} = \bar{x}$, from the case where $\underline{x} \neq \bar{x}$. We call the first situation a “degenerate cycle solution”, and the second one a “non-degenerate solution”.

3.2 Submodularity and Optimal Trajectories

In this paragraph, we introduce a submodularity assumption on the profit function π. Consider the following assumption.

Assumption 3 The function π is such that:

$$\pi(a, a - c) + \pi(b, b - d) \;\leq\; \pi(a, a - d) + \pi(b, b - c) \tag{11}$$

for every $d \leq c \leq b \leq a$.

Assumption 3 means that the profit generated by a big harvest in a large population, $\pi(a, a-d)$, augmented by the profit resulting from a small harvest in a medium sized population, $\pi(b, b-c)$, is greater than the sum of profits generated by two medium sized harvests, the first in a large population, $\pi(a, a-c)$, and the second in a medium sized population, $\pi(b, b-d)$.

If Assumption 2 holds as well, then in particular $\pi(b, 0) = 0$ and letting $c = b$ in (11), we have for all $d \leq b \leq a$:

$$\pi(a, a-b) + \pi(b, b-d) \;\leq\; \pi(a, a-d) \; . \tag{12}$$

In other words, one big harvest, $\pi(a, a-d)$, is better than two medium harvests, $\pi(a, a-b)$ and $\pi(b, b-d)$, reducing the population to the same level, i.e. d.

As far as the harvest is sold in a competitive market, the profit function is given as $\pi(x, I) = pI - c(x, I)$, where p is the price and $c(x, I)$ is the cost function. Then the above discussion translates in terms of costs (however Assumption 3 is a more general assumption linking together revenue and costs). Condition (11) reduces to the following property:

$$c(a, a-d) + c(b, b-c) \leq c(a, a-c) + c(b, b-d) \; .$$

The cost of a big harvest in a large population augmented by the cost of a small harvest in a medium population lower than the sum of the costs of two medium-sized harvests starting from the same large population a. Likewise, (12) becomes: $c(a, a-d) \leq c(a, a-b) + c(b, b-d)$. The cost of a big harvest is lower than the cost of two harvests starting and ending with the same population sizes, respectively a and d.

In some situations, we shall refer to a "strict" Assumption 3, meaning that:

$$\pi(a, a-c) + \pi(b, b-d) \;<\; \pi(a, a-d) + \pi(b, b-c) \tag{13}$$

for every $d < c < b < a$.

The following properties are well-known or easy to check.

Lemma 1 *Assume that π satisfies Assumption 3. Then:*

i) Let $g(x, y) = \pi(x, x-y)$ be defined for $0 \leq y \leq x \leq x_{sup}$. Then g is submodular on this domain.[7]

ii) If π has second-order derivatives, then inside the domain $\mathcal{D}$,

$$\pi_{xI} \;+\; \pi_{II} \geq 0 \; .$$

Conversely, this condition implies Assumption 3.

iii) If $\pi(x, I) = R(I)$, then R is convex. Conversely, if R is convex, Assumption 3 holds.

[7] A function $g(x, y)$ is submodular if for all a, b, c, d:

$$g(\min(a, b), \min(c, d)) + g(\max(a, b), \max(c, d)) \leq g(a, c) + g(b, d) \; .$$

Assumption 3 is weaker than both convexity of π with respect to harvest (which is equivalent to $\pi_{II} \geq 0$) or supermodularity of π (which is equivalent to $\pi_{xI} \geq 0$). The condition $\pi_{xI} + \pi_{II} \geq 0$ may hold if either of these properties holds, but does not imply them: it just implies that one of them locally holds.

Condition (12), with strict inequality, is classically required to insure the existence of optimal impulse control policies (see for instance Davis (1993)). But Assumption 3 cannot be reduced to condition (12), even under the requirement that $\pi(x, 0) = 0$. Indeed, consider for instance the case where $\pi(x, I) = R(I)$ for some function R. Then Assumption 3 says that R is convex (Lemma 1 *iii*)) whereas (12) says that R should be superadditive. It is known that some functions R with $R(0) = 0$ are superadditive without being convex. These conditions are therefore not equivalent.

3.3 Equivalence between (P) and (AP)

Now we are going to show the principal relation between problems (P) and (AP). The results of this section are partly based on the property that solutions to Problem (AP) turn out *not* to depend on x_0, as stated in Lemma 5, see Appendix A.3. Consequently, we can talk of solutions $(\underline{x}^*, \bar{x}^*)$ to the auxiliary problem (AP) independently of x_0. We then make the following assumption:

Assumption 4 The problem (AP) has a unique solution, denoted with $(\underline{x}^*, \bar{x}^*)$, which is such that $\underline{x}^* < \bar{x}^*$.

The transitory problem

Under Assumption 4, let us define the following optimization problem (TP), which formalizes the "Transitory Problem". The transitory part of a cyclical policy consists in a) letting the stock grow until some value x; b) harvesting from x down to y for $y \leq \bar{x}^*$; c) applying the cycle with the harvesting interval $[\underline{x}^*, \bar{x}^*]$ from then on. The question is how to choose the quantities x and y. The answer is given by the solutions of the following optimization problem:

$$\text{(TP)}: \quad \max_{\substack{x,y;\\ 0 \leq y \leq x \leq x_s\\ x_0 \leq x;\ y \leq \bar{x}^*}} \quad e^{-r\tau(x_0,x)} \left[\pi(x, x-y) + G(\underline{x}^*, \bar{x}^*, y)\right] \ .$$

The following theorem characterizes the solutions to the problem (P).

Theorem 2 *Assume that Assumptions 1–4 hold. Let $(x^*(x_0), y^*(x_0))$ solve the maximization problem (TP). Then the value function of (P) is:*

$$v(x_0) = \begin{cases} G(\underline{x}^*, \bar{x}^*, x_0) & if\ x_0 < \bar{x}^* \\ e^{-r\tau(x_0,x^*(x_0))} \big[\pi(x^*(x_0), x^*(x_0) - y^*(x_0)) \\ \qquad\qquad + G(\underline{x}^*, \bar{x}^*, y^*(x_0))\big] & if\ x_s \geq x_0 \geq \bar{x}^*. \end{cases} \tag{14}$$

Moreover there exists a solution of (P) which is cyclical and given by:

$$t_1 = \tau(x_0, \bar{x}^*), \quad \textit{and} \quad t_i = t_1 + (i-1)\tau(\underline{x}^*, \bar{x}^*), \;\; x_i = \bar{x}^*, \;\; I_i = \bar{x}^* - \underline{x}^*, \quad i \geq 1,$$

if $x_0 < \bar{x}^*$, *and*

$$t_1 = \tau(x_0, x^*(x_0)), \quad t_2 = \tau(y^*(x_0), \bar{x}^*), \quad t_i = t_2 + (i-2)\tau(\underline{x}^*, \bar{x}^*), \quad i \geq 2,$$

$$x_1 = x^*(x_0), \quad I_1 = x^*(x_0) - y^*(x_0), \quad x_i = \bar{x}^*, \quad I_i = \bar{x}^* - \underline{x}^*, \quad i \geq 2,$$

if $x_0 \geq \bar{x}^*$.

The proof of this result is given in Appendix A.3. The theorem states that any optimal cyclical policy has a cycle part with an harvesting interval $[\underline{x}^*, \bar{x}^*]$. It also describes the nature of the transitory part of optimal cyclical policies. In the case $x_0 < \bar{x}^*$, there is no transitory part, and the cycle is joined from the start. In the case $x_0 \geq \bar{x}^*$, the transitory part consists in letting the stock grow until $x^*(x_0)$, harvest it down to $y^*(x_0)$, then join the cycle. The typical form of optimal trajectories is illustrated in Figure 1.

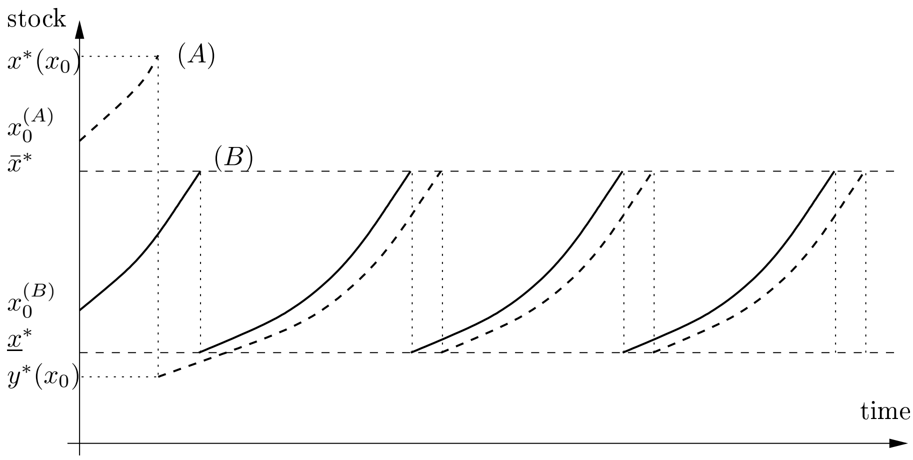

Fig. 1 Shape of the optimal trajectory, for $x_0 > \bar{x}^*$ (case (A)), and $x_0 \leq \bar{x}^*$ (case (B))

We can now state the following relation between problems (P) and (AP), the proof of which is provided in Appendix A.4.

Theorem 3 *Let Assumptions 1–3 hold. Then:*

i) If Assumption 4 holds as well, then (P) has a solution which is cyclical.
ii) If (P) has a solution, then (P) has a solution which is cyclical, and there exists a solution to problem (AP) when $0 \leq \underline{x} < \bar{x} \leq x_s$.
iii) If the solution of (AP) is on the boundary $\underline{x} = \bar{x} = x^*$, *then (P) has no solution.*

We have therefore shown that there exists a cyclical solution to our problem (P) if, and only if, the solution to the auxiliary problem (AP) is non-degenerate. In other words, the existence or not of cyclical solutions to (P) hinges on the fact that Assumption 4 holds or not. This question is addressed in the next section. Statement *iii*) of Theorem 3 results from the fact that, in this case, there is no policy in the set $\mathcal{F}_{x_0}$ which realizes the "sup" in Problem (P). However, the supremum does exist, and it can be shown that this value can be approached by a sequence of cyclical solutions.

We can come back to the interpretation of our central Assumption 3, in relation with the presence of cycles. Initiating the harvesting process is costly. Hence, cycles are optimal if resource managers can take advantage of some form of economies of scale: condition (12). This is the case, for instance, if the revenue function is convex, which is a consequence of Assumption 3 (Lemma 1 *iii*)) in the case of stock-independent costs. In addition, when π is linear in I, harvests and resource stocks are complementary (Lemma 1 *ii*)) and hence, any additional harvest, and resulting profits, can only be obtained by waiting and letting the resource recover, which comes at a cost.

In contrast to usual assumptions on the strict convexity of the profit function, Assumption 3 is more general as it covers the case of objective functions with multiple variables. It applies to convex-concave profit functions and is independent of any particular form of the dynamics $F(\cdot)$.

4 Optimal Cycles

We investigate now the problem of locating the solutions to Problem (AP). We have seen that solutions always exist, but they may be located in the interior, or on any of the boundaries $\underline{x} = 0$, $\bar{x} = x_s$ or the set $\underline{x} = \bar{x}$.

It turns out that ensuring the uniqueness of the solution is not an easy task, even with restrictive yet standard assumptions, as we argue in section 4.4. We therefore limit our discussion to conditions related to the submodularity Assumption 3. We begin in section 4.1 with necessary conditions for the existence of interior solutions and their interpretation. We study the case of strictly submodular functions in section 4.2, and the case of functions both submodular and supermodular in section 4.3.

4.1 Interior solutions

Necessary conditions for interior solutions to exist are given by the first order conditions of the auxiliary problem, which we provide as:

Lemma 2 *If $(\underline{x}, \bar{x})$ is a solution to the auxiliary problem (AP) with $0 < \underline{x} < \bar{x} < x_s$ (interior solution), then the first order conditions are given by:*

$$\pi_I \;=\; \frac{r}{F(\underline{x})} \frac{e^{-r\tau(\underline{x},\bar{x})}}{1 - e^{-r\tau(\underline{x},\bar{x})}} \; \pi(\bar{x}, \bar{x} - \underline{x}) \;, \tag{15}$$

$$\pi_x + \pi_I \;=\; \frac{r}{F(\bar{x})} \frac{1}{1 - e^{-r\tau(\underline{x},\bar{x})}} \; \pi(\bar{x}, \bar{x} - \underline{x}) \; . \tag{16}$$

By rearranging these conditions, we obtain the equivalent:

$$\pi_I \frac{F(\underline{x})}{r} \;=\; \frac{e^{-r\tau(\underline{x},\bar{x})}}{1 - e^{-r\tau(\underline{x},\bar{x})}} \; \pi(\bar{x}, \bar{x} - \underline{x}) \; , \tag{17}$$

$$\frac{d\pi}{dx} \;=\; \pi_x + \pi_I \;=\; \pi_I \frac{F(\underline{x})}{F(\bar{x}) e^{-r\tau(\underline{x},\bar{x})}} \; . \tag{18}$$

The first condition states that, at the optimum, the marginal gain from harvesting the resource, weighted with the growth potential at the new resource stock as compared to the discount rate, should equal the value of the remaining resource,[8] outcome of a maximized rotational harvest stream. The second condition states that the marginal gain derived from the stock effect is equal to the marginal gain from harvesting augmented by a correcting factor, which depends on the growth differential at the lower and upper limit of the rotational cycles, the latter being discounted over time. More precisely, the greater this growth differential, the greater the marginal gain due to the resource stock.

4.2 Strict submodularity of the gain function

In this section, we show that Assumption 3 in the strict sense, together with some technical assumptions, is sufficient to exclude degenerate cycle solutions to Problem (AP).

Going back to the definitions of Section 3.1, we have (see (10)):

$$G_d(x) \;:=\; G(x, x, x_0) \;=\; \frac{1}{r} \pi_I(x, 0) F(x) e^{-r\tau(x_0, x)} \; ,$$

where the choice of x_0 has no impact on the solution of the optimization problem, as we have seen. We can now state the result:

Proposition 1 *Assume that all maxima x_m of the function $G_d(x)$ are such that $0 < x_m < x_s$. If the function π has second-order derivatives and satisfies Assumption 3 in the strict sense* (13)*, then all solutions to Problem (AP) are non-degenerate.*

The proof is deferred to Appendix A.5.

[8] Which is called the site value in the forest economics literature.

4.3 Exact modularity

We now turn to the case where Assumption 3 holds with equality in Equation (11), which amounts to require that the function $\pi(x, x-y)$ be both sub- and supermodular. Using Lemma 1, it is not difficult to see that if π admits second-order derivatives, and given that $\pi(x, 0) = 0$, then it must be of the form:

$$\pi(\bar{x}, \bar{x} - \underline{x}) \;=\; \int_{\underline{x}}^{\bar{x}} \gamma(x)\; dx \tag{19}$$

for some integrable function $\gamma(\cdot)$ which is actually: $\gamma(x) = \pi_I(x, 0)$. We shall prove that, under moderate conditions, the problem (AP) does not admit non-degenerate solutions for such cost functions. In other words, solutions correspond necessarily to degenerate cycles.

In order to state this result formally, it is convenient to be sure that there is only one solution to the problem. For this reason, we add here several assumptions. We do not express them in terms of the primitives of the model, in order to keep them weaker than assumptions that would be put directly on the primitives. Indeed, although apparently restrictive, these assumptions appear to be satisfied in the examples we have studied using primitives from the literature.

Proposition 2 *Assume that the function $G_d(\cdot)$ is of class C^1, and is increasing, then decreasing for $x \in (0, x_s)$, with an unique maximum at x_m. Assume that G does not have a maximum at $\underline{x} = 0$, nor at $\bar{x} = x_s$. Then the solution of Problem (AP) is unique and given by $\underline{x} = \bar{x} = x_m$.*

The proof is deferred to Appendix A.6.

4.4 An example of multiple interior solutions to Problem (AP)

We provide in this section an example in which the data of the optimization problem satisfy usual assumptions (multiplicative separability, monotonicity, convexity), in which Property 3 holds, and yet Problem (AP) has two distinct interior solutions. It is constructed as follows. The standard logistic function $F(x) = x(1-x)$ is chosen as the growth function. It is concave. The gain function is chosen as $\pi(\underline{x}, I) = a(\bar{x}) \times I$, with, for some constant $A > 0$,

$$a(x) \;=\; 1 \;+\; \min\left\{ \frac{x}{100}, A \times (x - \frac{2}{3}) \right\} \;.$$

It can be easily verified that π satisfies Assumption 3, since the function a is strictly increasing. Finally, set $r = 0.01$. Numerical investigation then reveals that the function $G(\underline{x}, \bar{x}, x_0)$ corresponding to this data has two local maxima: one with $\bar{x} < 2/3$ and one with $\bar{x} > 2/3$. The local optimality of the first one results from the combination of a large growth rate with a small gain per cycle. Cycles are short for this solution. The second local optimum results from the combination of a smaller growth rate with a larger gain at each harvest. The two local maxima can be given the same value by setting the constant A to approximately 1.23.

5 Links between Impulse Control Models and Other Control Models

5.1 Comparison with Clark's Model

We may now establish a first link between our general impulse control model and the continuous control model, as proposed by Clark (1976).

Consider a solution of problem (AP) on the boundary $\underline{x} = \bar{x}$. The maximization problem becomes:

$$\max_{0 \le x \le x_s} G(x, x, x_0),$$

where G is given by (10). The first order condition for this problem is:

$$\pi_{Ix}(x,0)F(x) \; + \; \pi_I(x,0)[F'(x) - r] \; = \; 0 \; . \tag{20}$$

This condition coincides with the well-known marginal productivity rule of resource exploitation when $\pi_I(x,0)$ is the instantaneous profit function (see for example Clark (1976) or Clark and Munro (1975)). A solution to Equation (20) determines the steady state of the following Clark-like singular optimal control problem:

$$\text{(CP)} \qquad \max_{h(\cdot)} \int_0^{\infty} e^{-rt} \; \pi_I(x(t),0) \; h(t) \; dt \; ,$$

$$\dot{x} = F(x) - h,$$

for x_0 given and $0 \le h(t) \le h_{\max}$ for all t. This means that the conditions of a Clark-like steady state solution can also be triggered by the impulse control model that we propose.

5.2 Comparison with Dawid and Kopel's model

In this section, we show that Dawid and Kopel's model (1997) can be embedded within ours, through a judicious choice of the dynamics, the cost function and the discount rate. Then, we explain the correspondence between the results of Dawid and Kopel (1997) and ours.

5.2.1 Growth function and time span associated to the growth

The model of Dawid and Kopel is in discrete time. The population dynamics has the form:

$$x_{t+1} \;=\; f(x_t) - u_t \;=\; min[1, (1+\lambda)x_t] - u_t$$

with $x_t, u_t \geq 0 \; \forall t \geq 0$. We proceed by reproducing this behavior for our model. When no harvesting takes place, we must have: $\dot{x}(t) = F(x(t))$. Suppose:

$$F(x) \;=\; Ax \quad \text{if } x < x_s = 1 \quad \text{and} \quad F(x) = 1 - x \quad \text{if } x \geq 1.$$

It can be verified that this function satisfies Assumption 1.[9] Integrating the differential equation, we find that the stock evolves according to the following function:

$$x(t) \;=\; \phi(t, x_0) \;=\; \min(x_0 e^{At}, 1)\ .$$

In order to reproduce the dynamics of Dawid and Kopel's discrete-time model, we fix a time duration Δ, and set: $x_{t+1} = \phi(\Delta, x_t)$. The dynamics are equivalent when $f(x_t) = \phi(\Delta, x_t)$ for all x_t, which is the case when:

$$(1+\lambda)x_t \;=\; x_t e^{A\Delta}.$$

We deduce how the marginal growth factor A must be defined in terms of Dawid and Kopel's factor $1+\lambda$:

$$A \;=\; \frac{\log(1+\lambda)}{\Delta}.$$

5.2.2 Discounted benefits

For the undiscounted gains π, the correspondence with Dawid and Kopel's model is made by setting $\pi(x, I) = R(I)$. Note that for this particular form of the gain function, Condition (11) is equivalent to the convexity of R, according to Lemma 1 *iii*).

Next, the correspondence for discounting rates in both models is established as follows. The discrete-time discount factor being δ and the continuous-time discount rate being r, we should have: $\delta^t = e^{-rt\Delta}$, that is: $\log \delta = -r\Delta$. Finally, Dawid and Kopel's introduce a threshold quantity a defined as:

$$a \;=\; -\frac{\log \delta}{\log(1+\lambda)} \;=\; \frac{r\Delta}{A\Delta} \;=\; \frac{r}{A}\ .$$

We proceed with the definition of the function G which is the basis of the auxiliary problem (AP). Two cases must be considered: degenerate cycles or non-degenerate solutions.

[9] The value of $F(x)$ for $x > 1$ is arbitrarily chosen to that end.

Non-Degenerate: where $\underline{x} < \bar{x}$. In this case,

$$G(\underline{x}, \bar{x}, x_0) \;=\; \frac{R(\bar{x} - \underline{x})(\frac{x_0}{\bar{x}})^{\frac{r}{A}}}{1 - (\frac{\underline{x}}{\bar{x}})^{\frac{r}{A}}} \;=\; \frac{R(\bar{x} - \underline{x})(\frac{x_0}{\bar{x}})^{a}}{1 - (\frac{\underline{x}}{\bar{x}})^{a}}.$$

This expression holds even when $\bar{x} = x_s = 1$ and $\underline{x} = 0$.

Degenerate cycles: where $\underline{x} = \bar{x}$. Given that $\pi(x, I) = R(I)$, we have $\lim_{I \to 0} \pi_I(x, I) = R'(0)$, whence:

$$G(x, x, x_0) \;=\; R'(0)\; \frac{Ax}{r}\; \left(\frac{x_0}{x}\right)^{\frac{r}{A}} \;=\; R'(0)\; \frac{x_0^a}{a}\; x^{1-a}.$$

5.2.3 Relations with the Results by Dawid and Kopel

Dawid and Kopel define the elasticity of gains as the function:

$$\varepsilon(x) \;=\; \frac{R'(x)x}{R(x)}\;.$$

Through the analysis of the function G, the results of Dawid and Kopel can be reproduced, *modulo* the fact that decision instants are constrained in these results, and not for our model. For instance, if the elasticity of gains $\varepsilon(x)$ is larger than a for all x, it is optimal to defer harvesting until the resource reaches its maximal value. Dawid and Kopel obtain the same conclusion with the elasticity of gains averaged over the evolution of the population during one period. Inversely, when ε is smaller than a, immediate harvesting is optimal.

Other results of Dawid and Kopel address the question of whether immediate extinction is optimal or not. These results are reproduced by our analysis as well.

6 Conclusion

We have proposed an impulse control framework for the management of renewable resources which is general enough to include concave and convex gain functions, as well as stock dependent cost functions. The optimal management of the resource is expressed as optimization problem (P), the solution of which is shown to satisfy the dynamic programming principle. By introducing the class of "cyclical policies", we have reduced the solution of Problem (P) to the sequential solution of two static optimization problems with two variables each, which we can solve. With the help of the Auxiliary Problem, we can define the optimal cycle. With the Transitory Problem, we can describe the evolution from the initial stock to the cycle.

Central to our solution framework is the submodularity condition, which is necessary for the existence of cycles. This condition is more general than the strict convexity of the profit function, as it also covers the case of objective functions with multiple variables. Thus, the existence of economies of scale is only one possible condition for the occurrence of cycles, which depends on the more complex interaction between discounted gains, (stock dependent) cost functions and the population growth dynamics.

We have shown that our impulse control model can generate cyclical solutions and "degenerate" cyclical solutions which correspond to a smooth steady state solution. The economic and biological consequences of these two types of equilibria might be very different, especially if threshold values exist. For example, the cyclical solution may temporarily deplete the population underneath the level that would be desirable for the maintenance of the food-chain. These consequences are not taken into account in our model.

Our impulse control model can generate the steady state solution that Clark described for his one state variable model with a concave growth function. We can also replicate the cyclical policies described by Dawid and Kopel in a discrete-time framework with a quasi-linear growth function. This allows us to claim that our model is a "meta-model". The link between these models can be expressed through their responsiveness to the submodularity condition.

Recent bioeconomic models have strengthened the importance of uncertainty, for example linked to weather conditions or to the availability of stocks. Further research could include such uncertainty and also consider the manager's risk aversion in a similar impulse control framework. Econometric applications could help to check whether continuous or impulsive representations of the harvest decisions are more appropriate in practice, and how to specify growth and cost functions. Depending on the functional forms chosen, the optimal harvesting policies can then be defined within the above framework.

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A Appendix

A.1 Submodularity and Trajectories

We prove here trajectory comparison results which are a consequence of the submodularity Assumption 3. Before stating the results, we need some preliminary explanations.

Consider an impulse control policy ICP which is such that there exists i and j with $i < j$ and: $x_j - I_j \le x_i - I_i \le x_j \le x_i$, that is, overlapping harvests. Denote with $a = x_i$, $b = x_j$, $c = x_i - I_i$ and $d = x_j - I_j$. Let $\ell = j - i$ and $\delta t = t_j - t_i$. Consider the following two modifications of the reference policy ICP:

Policy A (copy a piece of trajectory from c to b):
- for $k < j$, $t^A_k = t_k$, $I^A_k = I_k$;
- for $k = j$, $t^A_j = t_j$, $I^A_j = b - c$;
- for $k > j$, $t^A_k = t_{k-\ell} + \delta t$, $I^A_k = I_{k-\ell}$.

Policy B (remove the piece of trajectory from c to b):
- for $k < i$, $t^B_k = t_k$, $I^B_k = I_k$;
- for $k = i$, $t^B_k = t_k$, $I^B_k = a - d$;
- for $k > i$, $t^B_k = t_{k+\ell} - \delta t$, $I^B_k = I_{k+\ell}$.

These policies can be visualized in Figure 2, which represents the evolution of the population under each of the three policies. The triangle represents the rest of the trajectory, which is the same for all three policies, except for a shift in time. The rectangle represents an arbitrary piece of trajectory, which can possibly exit the range $[b,c]$.[10]

The result is:

Lemma 3 *Consider an impulse control policy ICP which is such that there exists i and j with $i<j$ and: $x_j - I_j \le x_i - I_i \le x_j \le x_i$. Then:*

i) If Assumption 3 holds in the strict sense (13)*, then one of policies A or B constructed above yields strictly larger profits than ICP.*

ii) If Assumption 3 holds with equality in (11) *and if ICP is optimal, then policies A and B are optimal as well.*

Proof The discounted profits G associated with the original policy ICP can be written as:

$$G = V_0 \ + \ R_i \, \pi(a, a-c) \ + \ R_i \, V_1 \ + \ R_j \, \pi(b, b-d) \ + \ R_j \, V_d$$

where R_i and R_j are the discounts:

$$R_i \ = \ e^{-rt_i} \qquad R_j \ = \ e^{-rt_j} \ ,$$

and where V_0, V_1 and V_d are the current-value gains associated with the first part of the trajectory, and the pieces of the trajectory, respectively, in the intervals (t_i, t_j) and $(t_j, +\infty)$:

$$V_0 \ = \ \sum_{k=1}^{i-1} e^{-rt_k} \, \pi(x_k, I_k) \qquad V_1 \ = \ \sum_{k=i+1}^{j-1} e^{-r(t_k - t_i)} \, \pi(x_k, I_k)$$

$$V_d \ = \ \sum_{k=j+1}^{\infty} e^{-r(t_k - t_j)} \, \pi(x_k, I_k) \ .$$

The total discounted gains associated with policies A and B are:

$$\begin{aligned} G_A &= V_0 \ + \ R_i \, \pi(a, a-c) \ + \ R_i \, V_1 \ + \ R_j \, \pi(b, b-c) \ + \ R_j \, V_1 \\ &\quad + \ R_j \, \rho \, \pi(b, b-d) \ + \ R_j \, \rho \, V_d \\ G_B &= V_0 \ + \ R_i \, \pi(a, a-d) \ + \ R_i \, V_d \ , \end{aligned}$$

with $\rho = R_j/R_i = \exp(-r(t_j - t_i))$. Accordingly, modifications in profits implied by switching from the original policy to either A or B are:

$$\begin{aligned} G - G_A &= R_j \ \left(\pi(b, b-d) - \pi(b, b-c) + V_d - \rho\pi(b, b-d) - V_1 - \rho V_d \right) \\ G - G_B &= R_i \ \left(\pi(a, a-c) - \pi(a, a-d) + V_1 + \rho\pi(b, b-d) + \rho V_d - V_d \right) \ . \end{aligned}$$

As a consequence, we have the following identity:

$$\pi(a, a-c) + \pi(b, b-d) - \pi(a, a-d) - \pi(b, b-c) \ = \ \frac{1}{R_j}(G - G_A) \ + \ \frac{1}{R_i}(G - G_B) \ .$$

Under Assumption 3, the left-hand side is negative. If the inequality in (11) is strict, it is even strictly negative. This implies that one at least of G_A or G_B is strictly larger than G.

If equality holds (11) and the policy ICP is assumed to be optimal, then $G_A = G_B = G$ and policies A and B are optimal as well.

Consequences of Lemma 3 on the optimality of policies can be stated as:

Corollary 1 *Consider an impulse control policy ICP which is such that there exists i and j with $i<j$ and: $x_j - I_j \le x_i - I_i \le x_j \le x_i$. If Assumption 3 holds in the strict sense* (13)*, then ICP cannot be optimal.*

[10] The situation where $b = c$ is allowed, in which case the piece of trajectory may be empty. In that case, there is a double harvest at the same instant in time.

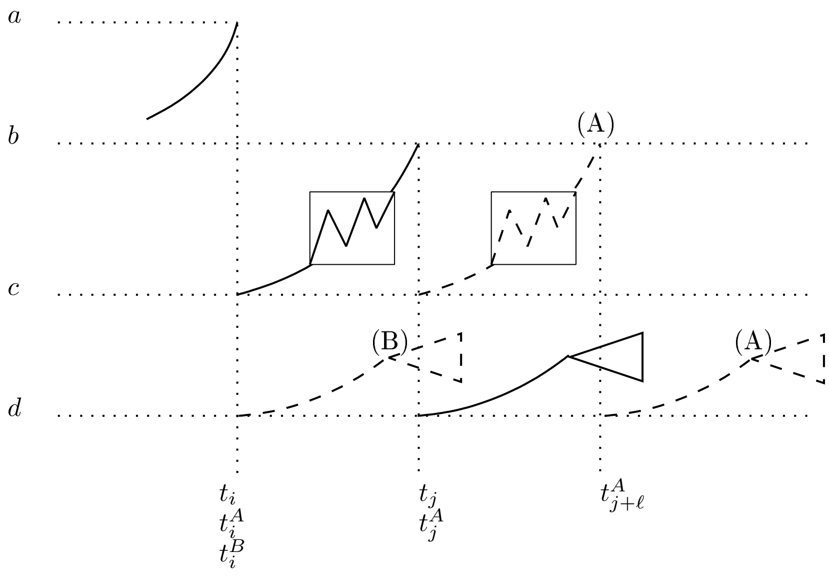

Fig. 2 The original policy and its modifications A and B

A.2 Dynamic Programming and Trajectories

In this appendix, we propose a technical result which is useful in a variety of situations. This comparison of trajectories is similar to Lemma 3 but it is provided by the application of the Dynamic Programming principle of Theorem 1.

Before stating the result, we need some preliminary explanations. Assume that a policy P is such that $x_{i+1} \geq x_i$. Let $\delta t = \tau(x_i - I_i, x_i)$. Consider the following modifications of the reference policy P:

Policy A (remove the harvesting at t_i)
 for $k < i$, $t_k^A = t_k$, $I_k^A = I_k$;
 for $k \geq i$, $t_k^A = t_{k-1} - \delta t$, $I_k^A = I_{k-1}$.
Policy B (copy once the harvesting occurring at t_i)
 for $k \leq i$, $t_k^B = t_k$, $I_k^B = I_k$;
 for $k > i$, $t_k^B = t_{k+1} + \delta t$, $I_k^B = I_{k+1}$.
Policy C (reproduce infinitely the harvesting occurring at t_i)
 for $k < i$, $t_k^C = t_k$, $I_k^C = I_k$;
 for $k \geq i$, $t_k^C = t_i + (k-i)\delta t$, $I_k^C = I_i$.

Assume now that the policy P is such that $x_{i+1} \in (x(t_{i-1}^+), x_i]$, where by convention, $t_{-1} = 0$ in the case $i = 1$. In that case, there exists a time $T = t_i - \tau(x_{i+1}, x_i)$ such that $x(T) = x_{i+1}$. As above, let $\delta t = \tau(x_i - I_i, x_i)$ and define the policies A, B and C exactly as above.

These policies are illustrated in Figure 3 a) in the first case, and b) in the second one.

We can now state the result:

Lemma 4 *Consider an impulse control policy P which is such that either $x_i \in (x(t_i^+), x_{i+1}]$ or $x_{i+1} \in (x(t_{i-1}^+), x_i]$ for some i. Then, the gain of policy P is smaller than that of policies A or C constructed above.*

Proof Assume first that policy P is such that $x_i \in (x(t_i^+), x_{i+1}]$, which implies $x_{i+1} \geq x_i$. Let G_P, G_A, G_B and G_C be the total profits for policies P, A, B and C. Denote with V_0 the

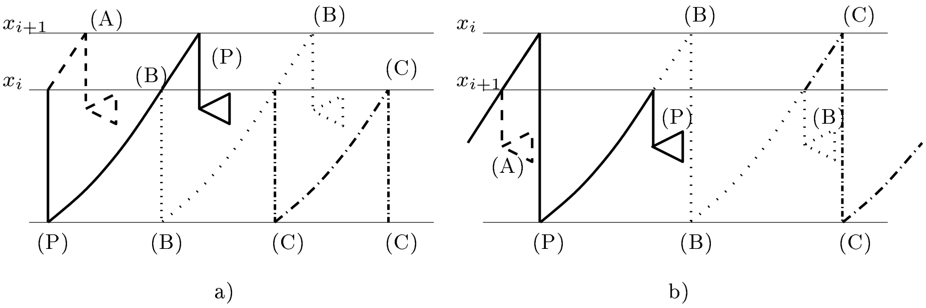

Fig. 3 The original policy ICP and its modifications A, B and C. The triangle represents the remainder of the trajectory, which is common to ICP, A and B, up to a shift in time.

current-value gains associated with the part of the trajectory before t_i (which is common to all these policies) and let $G_\pi = V_0 + e^{-rt_i}\tilde{G}_\pi$ for policies $\pi \in \{$ P, A, B, C $\}$. It is easy to see that

$$\begin{aligned}
\tilde{G}_P &= \pi(x_i, x_i - I_i) + e^{-r\delta t}\tilde{G}_A \\
\tilde{G}_B &= \pi(x_i, x_i - I_i) + e^{-r\delta t}\tilde{G}_P \\
\tilde{G}_C &= \pi(x_i, x_i - I_i) + e^{-r\delta t}\tilde{G}_C \ .
\end{aligned}$$

Consequently, we have the identity: $\tilde{G}_P - \tilde{G}_B = e^{-r\delta t}(\tilde{G}_A - \tilde{G}_P)$. This implies that $\tilde{G}_P \leq \max(\tilde{G}_A, \tilde{G}_B)$. Next, if we have $\tilde{G}_P \leq \tilde{G}_B$, then we have $\tilde{G}_B \leq \pi(x_i, x_i - I_i) + e^{-r\delta t}\tilde{G}_B$ so that:

$$\tilde{G}_B \ \leq \ \frac{\pi(x_i, x_i - I_i)}{1 - e^{-r\delta t}} \ = \ \tilde{G}_C \ .$$

This proves the statement.

Consider now the case $x_{i+1} \in (x(t_{i-1}^+), x_i]$. As argued above, the time $T = t_i - \tau(x_{i+1}, x_i)$ is such that $x(T) = x_{i+1}$. Let $\tilde{G}_i$ be the current-value gains of the different policies at time $t = T$. It is clear that:

$$\begin{aligned}
\tilde{G}_P &= e^{-r(t_i - T)}\pi(x_i, x_i - I_i) + e^{-r\delta t}\tilde{G}_A \\
\tilde{G}_B &= e^{-r(t_i - T)}\pi(x_i, x_i - I_i) + e^{-r\delta t}\tilde{G}_P \\
\tilde{G}_C &= e^{-r(t_i - T)}\pi(x_i, x_i - I_i) + e^{-r\delta t}\tilde{G}_C \ .
\end{aligned}$$

As a result, we have the same identity: $\tilde{G}_P - \tilde{G}_B = e^{-r\delta t}(\tilde{G}_A - \tilde{G}_P)$, and the rest of the previous reasoning applies.

A.3 Proof of Theorem 2

The proof is separated into two cases. If x_0 is "small enough", the proof is provided by trajectory comparison arguments. For the case of "large" x_0, the proof consists in embedding the optimization problems (AP) and (TP) into a more general optimization problem, then solving this more general problem. The solution turns out to be provided by (AP) and (TP), and satisfy the dynamic programming equation.

Throughout the rest of this section, Assumptions 1 and 2 hold, so that the function G is well defined, and Assumption 4 is assumed to hold as well, so that the optimal values for (AP), $\underline{x}^*$ and $\bar{x}^*$, are well defined.

Let $w(\cdot)$ be defined, as in (14), as:

$$w(x) = \begin{cases} G(\underline{x}^*, \bar{x}^*, x) & \text{if } x < \bar{x}^* \\ e^{-r\tau(x,x^*(x))}\left[\pi(x^*(x), x^*(x) - y^*(x)) + G(\underline{x}^*, \bar{x}^*, y^*(x))\right] & \text{if } x_{sup} \geq x \geq \bar{x}^* \end{cases} \tag{21}$$

where $(x^*(x), y^*(x))$ is any solution of the problem (TP) with initial population $x_0 = x$.

The following result will be useful for the proof. Consider problem (AP). Its solution does not depend on the initial stock value x_0:

Lemma 5 *Assume that $(\underline{x}^*, \bar{x}^*)$ solves (AP) for some value of $x_s > x_0 > 0$. Then it solves (AP) for every value of x_0.*

Proof The result follows from the fact that for all x_0, x_1:

$$G(\underline{x}, \bar{x}, x_0) \;=\; e^{-r\tau(x_0,x_1)}\; G(\underline{x}, \bar{x}, x_1)\ .$$

Therefore the two functions are proportional, with a proportionality factor which is strictly positive if $0 < x_0 < x_s$ and $0 < x_1 < x_s$. The problems (AP) for x_0 and (AP) for x_1 have therefore the same solutions. If $x_1 = 0$, or if $x_1 = x_s$ and $\lim_{y \uparrow x_s} \tau(x, y) = +\infty$, then $G = 0$ and any $(\underline{x}^*, \bar{x}^*)$ maximizes it.

A.3.1 Proof for $x_0 < \bar{x}^*$

Lemma 6 *If Assumptions 3 and 4 hold, then the function $w(x_0)$ solves the dynamic programming equation (7) for all $x_0 < \bar{x}^*$.*

According to Theorem 1, the value function of problem (P) verifies:

$$v(x) = \max_{\substack{t \geq 0 \\ 0 \leq y \leq \phi(t,x)}} e^{-rt}\left[\pi(\phi(t,x), \phi(t,x) - y) + v(y)\right] \tag{22}$$

$$= \max\Bigg\{ \max_{0 \leq y \leq x} \left[\pi(x, x-y) + v(y)\right]\ , \tag{23}$$

$$\max_{\substack{\bar{x}, y; \\ x < \bar{x} \leq x_s \\ 0 \leq y \leq \bar{x}}} e^{-r\tau(x,\bar{x})}\left[\pi(\bar{x}, \bar{x} - y) + v(y)\right] \Bigg\}\ . \tag{24}$$

This breakdown is obtained by separating the case $t = 0$ (expression (23)) from the case $t > 0$, and performing the change of variable $t = \tau(x, \bar{x})$ in (24). This change of variable maps the time interval $t \in (0, +\infty)$ to the interval on populations $\bar{x} \in (x, x_s)$ or $\bar{x} \in (x, x_s]$, depending on whether $\tau(x, y)$ diverges or not when $x \downarrow 0$.

We must show that the function $w(x)$, defined in (21), is a solution of Equation (22).

By assumption, $x < \bar{x}^*$. Replacing $v(y)$ by its value in (22), the right-hand side can be written as $M = \max\{M_1, M_2, M_3\}$ where:

$$M_1 = \max_{0 \leq y \leq x} \left[\pi(x, x-y) + G(\underline{x}^*, \bar{x}^*, y)\right]\ , \tag{25}$$

$$M_2 = \max_{\substack{x < \bar{x} \leq x_s \\ 0 \leq y < \bar{x}^*}} e^{-r\tau(x,\bar{x})}\left[\pi(\bar{x}, \bar{x} - y) + G(\underline{x}^*, \bar{x}^*, y)\right]\ , \tag{26}$$

$$M_3 = \max_{\substack{x < \bar{x} \leq x_s \\ \bar{x}^* \leq y \leq \bar{x}}} e^{-r\tau(x,\bar{x})}\Big[\pi(\bar{x}, \bar{x} - y) \tag{27}$$

$$+ e^{-r\tau(y,x^*(y))}\left[\pi(x^*(y), x^*(y) - y^*(y)) + G(\underline{x}^*, \bar{x}^*, y^*(y))\right]\Big]\ .$$

We recognize in the term (26) the problem (TP). We prove first that this is the largest of the three. Consider, for some $y = y_0$, the value in brackets in (27). It corresponds to a policy

P with two harvests $\bar{x} \to y_0$ and $x^*(y_0) \to y^*(y_0)$. Two cases may happen, according to which of $\bar{x}$ and $x^*(y_0)$ is the largest.

Case $\bar{x} \geq x^*(y_0)$: in this case, these two harvests are overlapping (since $y^*(y_0) < \bar{x}^* \leq y_0$), in which case Lemma 3 applies. The policy P is dominated by at least one of two modifications. If the dominating policy is the one excluding the second harvest, then its value is present in (26) when y has the value $y^*(y_0)$. If the dominating policy is the one with an additional harvest, then it is obvious (see for instance the proof of Lemma 4) that the policy with a cyclical harvesting with interval $[\bar{x}^*, y]$ is even better. But this policy provides a gain equal to $\pi(\bar{x}, \bar{x} - y) + e^{-\tau(y,\bar{x}^*)} G(y, \bar{x}^*, y) \leq \pi(\bar{x}, \bar{x} - y) + e^{-\tau(y,\bar{x}^*)} G(\underline{x}^*, \bar{x}^*, y)$. Policy P is therefore again dominated by some policy represented in (26).

Case $\bar{x} < x^*(y_0)$: in this case, Lemma 4 applies, and policy P is dominated by at least one of two modifications. Either the dominating policy is the modification "A" without a second harvest: its gain is one of the values in (26). Or the dominating policy is the one with a cyclical harvesting. The reasoning above then applies and there is a value in (26) which dominates the value in (27). We have shown that (27) is smaller than (26).

Next, we show that (25) is dominated by (26). Each y in (25) corresponds to some policy P_y for which the two first harvests are $x \to y$ and $\bar{x}^* \to \underline{x}$. Since x is smaller than $\bar{x}^*$, we are once more in the situation of Lemma 4. The policy P_y is therefore dominated: either by the policy A which consists in directly applying the cycle with interval $[\underline{x}^*, \bar{x}^*]$, or by the cyclical policy with interval $[y, x]$. This one is in turn dominated by the cyclical policy A according to Assumption 4. In both cases, P_y is dominated by C. Since the gain associated with C is present in (26) (with $\bar{x} = \bar{x}^*$ and $y = \underline{x}^*$), the term in (25) is dominated by the term in (26).

At this stage, we have proved that (26) dominates the two other terms, so that:

$$M \;=\; \max_{\substack{x < \bar{x} \leq x_s \\ 0 \leq y < \bar{x}^*}} e^{-r\tau(x,\bar{x})} \left[\pi(\bar{x}, \bar{x} - y) + G(\underline{x}^*, \bar{x}^*, y) \right] \;.$$

It now remains to be proved that the maximum in the right-hand side is reached at $\bar{x} = \bar{x}^*$ and $y = \underline{x}^*$. Each value of the right-hand side is the gain of some policy P for which the two first harvests are $x_1 = \bar{x}$ and $x_2 = \bar{x}^*$. Whether $\bar{x} < \bar{x}^*$ or $\bar{x} > \bar{x}^*$, the application of Lemma 4 implies that P is dominated: either by policy "A" which has the value $G(\underline{x}^*, \bar{x}^*, x)$, or by policy "C" which has the value $G(y, \bar{x}, x) < G(\underline{x}^*, \bar{x}^*, x)$ by Assumption 4 and Lemma 5.

The value of M is readily seen to be $e^{-r\tau(x,\bar{x}^*)} G(\underline{x}^*, \bar{x}^*, \bar{x}^*) = G(\underline{x}^*, \bar{x}^*, x) = w(x)$. The function w solves the Bellman equation for $x < \bar{x}^*$.

A.3.2 Proof for $x_0 \geq \bar{x}^*$

Lemma 7 *If Assumptions 3 and 4 hold, then the function $w(x_0)$ solves the dynamic programming equation for all $x_s \geq x_0 \geq \bar{x}^*$.*

Proof Replacing $v(y)$ by its value in (22), the right-hand side, say M', can be written as the maximum of the four terms:

$$\max_{0\le y<\bar{x}^*} [\pi(x_0, x_0 - y) + G(\underline{x}^*, \bar{x}^*, y)] \ , \tag{28}$$

$$\max_{\bar{x}^*\le y\le x_0} \Big[\pi(x_0, x_0 - y) \tag{29}$$

$$+ e^{-r\tau(y,x^*(y))} [\pi(x^*(y), x^*(y) - y^*(y)) + G(\underline{x}^*, \bar{x}^*, y^*(y))] \Big] \ ,$$

$$\max_{\substack{x_0<\bar{x}\le x_s\\ 0\le y<\bar{x}^*}} e^{-r\tau(x_0,\bar{x})} [\pi(\bar{x}, \bar{x} - y) + G(\underline{x}^*, \bar{x}^*, y)] \ , \tag{30}$$

$$\max_{\substack{x_0<\bar{x}\le x_s\\ \bar{x}^*\le y\le \bar{x}}} e^{-r\tau(x_0,\bar{x})} \Big[\pi(\bar{x}, \bar{x} - y) \tag{31}$$

$$+ e^{-r\tau(y,x^*(y))} [\pi(x^*(y), x^*(y) - y^*(y)) + G(\underline{x}^*, \bar{x}^*, y^*(y))] \Big] \ .$$

Following the reasoning in proof of Lemma 6, the terms (29) and (31) are respectively dominated by (28) and (30). There remains:

$$\begin{aligned} M' &= \max\Big\{ \max_{0\le y\le \bar{x}^*} [\pi(x_0, x_0 - y) + G(\underline{x}^*, \bar{x}^*, y)] \ , \\ &\qquad \max_{\substack{x_0<\bar{x}\le x_s\\ 0\le y\le \bar{x}^*}} e^{-r\tau(x_0,\bar{x})} [\pi(\bar{x}, \bar{x} - y) + G(\underline{x}^*, \bar{x}^*, y)] \Big\} \\ &= \max_{\substack{x_0\le\bar{x}\le x_s\\ 0\le y\le \bar{x}^*}} e^{-r\tau(x_0,\bar{x})} [\pi(\bar{x}, \bar{x} - y) \ + \ G(\underline{x}^*, \bar{x}^*, y)] \ . \end{aligned}$$

This is the definition of Problem (TP). The solution is therefore $(x^*(x_0), y^*(x_0))$, which concludes the proof.

A.4 Proof of Theorem 3

The statement *i*) of Theorem 3 is a direct consequence of Theorem 2.

For statement *ii*), we need the following result, which is a corollary of Assumption 3 and Lemma 3.

Lemma 8 *If Assumption 3 holds, then for every solution to problem (P) which is not cyclical, there exists a cyclical solution with the same value.*

Proof It is first necessary to characterize what a non-cyclical solution may be. From the definition of cyclical policies in Section 3.1, it can be seen by inspection (see also Figure 1) that the set of possible values for the population $x(t)$ is made of at most two intervals included in $[0, x_s]$, and that every single value a) is either reached once only, b) or is reached an infinite number of times according to a periodic sequence $s_1, s_1 + T, s_1 + 2T, \ldots$ for some $T > 0$, c) or is 0. A solution which is not cyclical would therefore: *i*) either reach population values in more than three disjoint intervals, *ii*) or reach some value $v \neq 0$ a number of times which is neither 1 nor infinity, *iii*) or reach some value $v \neq 0$ according to a sequence of instants which is not periodic.

The first step is to exclude non-cyclical solutions to (P) which are such that $x(s) = x(t)$ for some $s < t$. For such a policy (A), consider the smallest such t. Let (B) be the policy which consists in performing the same harvests as (A) up to time t, next applying the optimal cyclical policy with initial population $x(t)$ but shifted in time by t units. The values reached by policy (B) are reached either once or an infinite number of times at periodic intervals. As a consequence of Theorem 1, the value function of policy (B) is the same as (A). Therefore, a policy which is such that *ii*) or *iii*) can be replaced by a cyclical one.

The second step is to eliminate policies of type *i*). For such policies, there exists some $i < j$ and a sequence of values $a > b \geq c > d$, such that for some i, $x_i = a$, $I_i = a - c$, and $x_j = b$, $I_j = b - d$. According to Lemma 3, such a policy cannot be optimal if Assumption 3 strictly holds. In the other case, the policy can be replaced with another policy with the same total profit but with one less harvest. If this policy is not cyclical, an induction is applied to construct a cyclical policy which produces the same profit as the original one.

According to this lemma, we know that we can restrict our attention to cyclical solutions of (P). Such solutions are characterized by Theorem 2. Their cyclical part is given by an harvesting interval $[\underline{x}^*, \bar{x}^*]$ which is necessarily an interior solution of (AP).

Finally, statement *iii*) is a consequence of statement *ii*): if (P) had a solution, the solution of (AP) would be a non-degenerate solution.

A.5 Proof of Proposition 1

Proof First, observe that the identity $\pi(x, 0) = 0$ implies that for all x, $\pi_x(x, 0) = 0$ and $\pi_{xx}(x, 0) = 0$. Taking this into account and developing G in a neighborhood of the point $\underline{x} = \bar{x} = x$ using a Taylor series, we obtain:

$$G(x + h, x + k, x_0) \;\cong\; G(x, x, x_0) \;+\; \frac{F(x)}{r} e^{-r\tau(x_0,x)} B(x, h, k), \tag{32}$$

where, introducing $\epsilon = h - k$,

$$B(x, h, k) = \frac{\epsilon}{2}\left[\pi_{II}(x, 0) - \frac{r - F'(x)}{F(x)}\;\pi_I(x, 0)\right] \;+\; h\left[\frac{r - F'(x)}{F(x)}\;\pi_I(x, 0) + \pi_{xI}(x, 0)\right] \;.$$

Any maximum x_m of the function $G(x, x, x_0)$ satisfies the fist-order condition $B(x_m, h, h) = 0$ for sufficiently small values of h. Therefore,

$$0 \;=\; \frac{r - F'(x_m)}{F(x_m)}\;\pi_I(x_m, 0) + \pi_{xI}(x_m, 0) \;.$$

Consequently,

$$\begin{aligned} B(x_m, h, k) &= \frac{\epsilon}{2}\left[\pi_{II}(x_m, 0) - \frac{r - F'(x_m)}{F(x_m)}\;\pi_I(x_m, 0)\right] \\ &= \frac{\epsilon}{2}\;(\pi_{II} + \pi_{xI})(x_m, 0) \;. \end{aligned}$$

From Lemma 1 *ii*), adapted to the strict inequality in (13), we know that $(\pi_{II}+\pi_{xI})(x_m, 0) > 0$. Therefore, for any small deviations h and $\epsilon > 0$ towards the interior of the domain, $B(x_m, h, h - \varepsilon) > 0$, and we conclude that there are values of $G(\underline{x}, \bar{x}, x_0)$ which are larger than $G(x_m, x_m, x_0)$. The solution to (AP) thus cannot be such that $\underline{x} = \bar{x}$, so that the optimal cycle is non-degenerate.

A.6 Proof of Proposition 2

First of all, we can rule out solutions of (AP) with $\underline{x} = 0$, or $\bar{x} = x_s$, by assumption.

Next, we rule out interior solutions. According to Lemma 2, specialized to integral gain functions, an interior solution $0 < \underline{x} < \bar{x} < x_s$ should satisfy the system of equations:

$$\gamma(\underline{x}) = \frac{r}{F(\underline{x})} \frac{e^{-r\tau(\underline{x},\bar{x})}}{1 - e^{-r\tau(\underline{x},\bar{x})}} \int_{\underline{x}}^{\bar{x}} \gamma(u)\ du \tag{33}$$

$$\gamma(\bar{x}) = \frac{r}{F(\bar{x})} \frac{1}{1 - e^{-r\tau(\underline{x},\bar{x})}} \int_{\underline{x}}^{\bar{x}} \gamma(u)\ du\ . \tag{34}$$

Here, the constant x_0 is still arbitrary. It is easily seen that the system of equations (33)–(34) is equivalent to (35)–(36), where:

$$\gamma(\underline{x})F(\underline{x})e^{-r\tau(x_0,\underline{x})} = \gamma(\bar{x})F(\bar{x})e^{-r\tau(x_0,\bar{x})} \tag{35}$$

$$\gamma(\underline{x})F(\underline{x}) - r\int_{x_0}^{\underline{x}} \gamma(u)\ du = \gamma(\bar{x})F(\bar{x}) - r\int_{x_0}^{\bar{x}} \gamma(u)\ du\ . \tag{36}$$

Condition (35) is in turn equivalent to $G_d(\underline{x}) = G_d(\bar{x})$, while (36) can be written as $\varphi(\underline{x}) = \varphi(\bar{x})$, with the definition:

$$\varphi(x) = \frac{1}{r}\gamma(x)F(x) - \int_{x_0}^{x} \gamma(u)\ du\ .$$

It is convenient here to pick as x_0 the value x_m provided by the hypothesis. For this choice, we have $G_d(x_m) = \varphi(x_m) = \gamma(x_m)F(x_m)/r$. We now prove that $x < x_m$, then $\varphi(x) < G_d(x)$ and if $x > x_m$, then $\varphi(x) > G_d(x)$. Indeed, differentiation of φ readily gives:

$$\varphi'(x) = G_d'(x)\ e^{r\tau(x_m,x)}\ .$$

The value of $e^{-r\tau(x_m,x)}$ is positive and larger than 1 if $x_m > x$, and is smaller than 1 if $x_m < x$. But according to the hypothesis, $G_d'(x) \geq 0$ if $x_m > x$ and $G_d'(x) \leq 0$ if $x_m < x$. All these facts finally imply that $\varphi'(x) \leq G_d'(x)$ for all x. This in turn implies the property stated above.

But then for any $\underline{x} < \bar{x}$ such that $G_d(\underline{x}) = G_d(\bar{x})$, the hypothesis implies $\underline{x} < x_m < \bar{x}$. Therefore, we have:

$$\varphi(\underline{x}) > G_d(\underline{x}) = G_d(\bar{x}) > \varphi(\bar{x})\ ,$$

which excludes the possibility that $\varphi(\underline{x}) = \varphi(\bar{x})$. We have therefore proved that no interior solution exists.

There remain the solutions on the boundary $\underline{x} = \bar{x}$. Again appealing to the hypothesis, the maximum on this boundary, and therefore the global maximum, is $\underline{x} = \bar{x} = x_m$. This concludes the proof.